AD-A104 083 CARNEGIE-MELLON UNIV PITTSBURGH PA MANAGEMENT SCIENC--ETC F/G 15/5
STRONG PLANNING AND FORECAST HORIZONS FOR A MODEL WITH SIMULTAN--ETC(U)
JUN 81 G L THOMPSON, S P SETHI, J TENG N00014-79-C-0340
UNCLASSIFIED MSRR-476 NL

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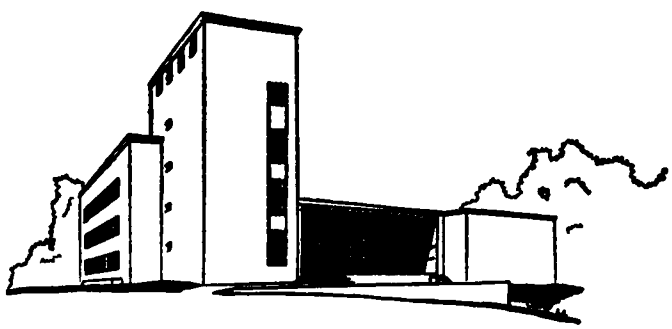

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(14) MSRR-476 (1)
W.P. No. 55-80-81

(6) Management Sciences Research Report No. 476

(6) ~~STRONG~~ ~~PLANNING~~ AND ~~FORECAST~~ ~~HORIZONS~~
FOR A ~~MODEL~~ WITH ~~SIMULTANEOUS~~ ~~PRICE~~ AND ~~PRODUCTION~~ ~~DECISIONS~~.

by

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(11) June 1981 (12) 42

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Abstract

Strong Planning and Forecast Horizons for a Model with Simultaneous Price and Production Decisions

by

G.L. Thompson, S.P. Sethi, and J.T. Teng

In this paper, we have solved a general inventory model with simultaneous price and production decisions. Both linear and non-linear (strictly convex) production cost cases are treated. Upper and lower bounds are imposed on state as well as control variables. The problem is solved by using the Lagrangian form of the maximum principle. Strong planning and strong forecast horizons are obtained. These arise when the state variable reaches its upper or lower bound. The existence of these horizons permits the decomposition of the whole problem into a set of smaller problems, which can be solved separately, and their solutions put together to form a complete solution to the problem. Finally, we derive a forward branch and bound algorithm to solve the problem. the algorithm is illustrated with a simple example.

Key Words

Strong planning horizon

optimal control theory

price-production-inventory model

forward branch and bound algorithm

1. Introduction

Many methods have been developed to solve the problem of production scheduling over future periods of time in order to satisfy the given amounts required of a certain commodity in each of these periods. Such methods are based on a wide range of production-inventory models--from elementary deterministic ones to sophisticated stochastic models with time-varying parameters. Historically, applications of optimization methods to production and inventory problems date back at least to the classical EOQ model and the lot size formula. The EOQ model is essentially a static model in the sense that it assumes the demand to be constant and only a stationary solution is sought. A dynamic version of the lot size model in which the assumption of a steady-state demand rate was dropped was analyzed by Wagner and Whitin [13]. The solution methodology used there was dynamic programming.

In some dynamic problems it is possible to show that the optimal decisions made during an initial positive time interval are either partially or wholly independent of the data from some future time onwards. In such cases, a forecast of the future data needs to be made only as far as that time to make optimal decisions in the initial time interval. The intial time interval is called the <u>planning horizon</u> and the time up to which data is required to make the optimal decisions during the planning horizon is called the <u>forecast horizon</u>. Whenever they exist, these horizons naturally decompose the problem into a series of smaller problems.

If the optimal decisions during the planning horison are completely independent of the data beyond the forecast forizon, then the former is called a strong planning horizon and the latter is called a strong forecast horizon. If, on the other hand, some mild restrictions on the data after the forecast horizon are required in order that to keep the optimal decisions during the planning horizon unaffected. then it is called a weak forecast horizon, and the corresponding planning horizon is called week planning horizon. For example, Modigliani and Hohn [5] show that a planning horizon $[0,t^*]$ in their production planning model, the corresponding strong forecast horizon is always the overall horizon $[0,T]$, but the information required on $[t^*,T]$ is just the accumulated demand rather than the complete demand schedule for each instant of time in that interval. In the case, the weak planning horizon and the weak forecast horizon are identical. Of course, a weak planning horizon does not reduce the interval over which forecasts are needed, but reduces the informational requirement within this interval.

Earlier planning horizon results for production planning, obtained by using dynamic programming and variational arguments, have been given by Kunreuther and Morton [2,3], Modigliani and Hohn [5], and Wagner and Whitin [13]. Other horizon results, derived by using optimal control theory, were given by Kleindorfer and Lieber [1], Lieber [4], Morton [6], Pekelman [7,8], Teng, Thompson, and Sethi [10], and Vanthienen [12].

In this paper, the problem of determining simultaneously the price and production schedule of a firm over a finite horizon T is solved for a given time dependent demand function. Both linear and non-linear (strictly convex) production cost cases are treated. Upper and lower bounds are imposed on state as well as control variables. The problem is solved by using the Lagrangian form of the maximum principle. Strong planning and strong forecast horizons are obtained. These arise when the state variable reaches its upper or lower bound. The existence of these horizons permits the decomposition of the whole problem into a set of smaller problems, which can be solved separately, and their solutions put together to form a complete solution to the problem. Finally, we derive a forward branch and bound algorithm to solve the linear case. The forward algorithm for the convex case is similar.

2. The Linear Production Cost Case

In this section, we will find the optimal price $p^*(t)$ and production rate $u^*(t)$ of a monopolist who faces a differentiable demand curve on the interval [0.T]. Suppose the demand curve is $D(t) = a(t)-bp(t)$ and the linear production cost is $cu(t)$, the linear inventory cost is $hI(t)$, the maximum production rate is $\bar{u}$, and the warehouse constraint is $I(t) \leq W$. The resulting problem is that of maximizing the following expression:

$$J_c = \int_0^T \{p(t)[a(t)-bp(t)]-hI(t)-cu(t)\}dt \qquad (1)$$

subject to

$$[\lambda] \quad \dot{I}(t) = u(t)-[a(t)-bp(t)], \; I(0) = I_0 \leq W \tag{2}$$

$$\begin{aligned} &[\rho_1] \; I(t) \geq I(t) \geq 0, \; [\rho_2] I(t) \leq W, \; [\mu_1] u(t) \geq 0, \\ &[\mu_2] \; u(t) \leq \bar{u}, \; [\eta_1] \; p(t) \geq 0, \; [\eta_2] p(t) \leq a(t)/b, \end{aligned} \tag{3}$$

where $h,c,b,\bar{u}$ and W are positive constants and $c \leq a(t)/b$ for all t. Note that $a(t)/b$ represents the maximum price, which is required to exceed the production cost (otherwise, no production will take place). The function $\lambda(t)$ is the adjoint variable of (2), and $\rho_1(t)$, $\rho_2(t)$, $\mu_1(t)$, $\mu_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are the Lagrange variables of the corresponding constraints. A dot above a variable denotes the first derivative with respect to time.

This model is similar to the model due to Pekelman [7] except that the production cost here is linear, and production and warehouse upper bound constraints have been added. In Section 3 we consider the same model with convex production cost, which is a generalization of Pekelman's.

2.1 The Necessary Conditions for an Optimal Solution

The Hamiltonian function of Problem (1) is

$$H = pa-p^2b-hI-cu-\lambda(u-a+bp) \tag{4}$$

which is linear in u and quadratic in p so that the optimal production rate $u^*(t)$ is <u>bang-bang</u> and the optimal price $p^*(t)$ is a <u>sat function</u>. For this problem the Lagrangian is

$$L=H+\rho_1(u-a+bp)+\rho_2(a-bp-u)+\mu_1 u+\mu_2(\bar{u}-u)+\eta_1 p+\eta_2(a/b-p) \tag{5}$$

The following necessary conditions hold, see [9], for an optimal solution

$$\frac{\partial L}{\partial u}=0=-c+\lambda+\rho_1-\rho_2+\mu_1-\mu_2, \text{ or}$$

$$u^*=\begin{cases} 0 & \text{if } \lambda+\rho_1-\rho_2<c \\ \text{undefined} & \text{if } \lambda+\rho_1-\rho_2=c \\ \bar{u} & \text{if } \lambda+\rho_1-\rho_2>c; \end{cases} \tag{6}$$

$$\frac{\partial L}{\partial p}=0=a-2bp+(\lambda_1+\rho_1-\rho_2)b+\eta_1-\eta_2, \text{ or}$$

$$p^*=\begin{cases} 0 & \text{if } \lambda+\rho_1-\rho_2\leq -a/b \\ (a/b+\lambda+\rho_1-\rho_2)/2 & \text{if } -a/b<\lambda+\rho_1-\rho_2<a/b \\ a/b & \text{if } a/b\leq\lambda+\rho_1-\rho_2; \end{cases} \tag{7}$$

the adjoint equation satisfies

$$\dot{\lambda}=-\partial L/\partial I=h \tag{8}$$

and the transversality conditions are

$$\lambda(T)\geq 0 \text{ and } \lambda(T)I(T)=0; \tag{9}$$

the complementarity and nonnegativity conditions are

$$\left\{\begin{array}{l} \rho_1 I, \rho_1 \dot{I}, \rho_2(W-I), \rho_2 \dot{I}, \mu_1 u, \mu_2(\bar{u}-u), \eta_1 p, \eta_2(a/b-p)=0 \\ \\ \text{and } \rho_1, \rho_2, \mu_1, \mu_2, \eta_1, \eta_2 \geq 0; \end{array}\right. \qquad (10)$$

the variable $I(t)$ is continuous, λ is continuous except possibly at an entry time or an exit time to the boundary conditions $I(t)=0$ or $I(t)=W$, $\lambda+\rho_1-\rho_2$ is continuous everywhere, and moreover $\dot{\rho}_1 \leq 0$ and $\dot{\rho}_2 \leq 0$. (11)

Note that t_{iw} is an <u>initial</u> or <u>entry</u> time to $I(t) = W$ if $I(t_{iw}) < W$, and $I(t_{iw})=I(t_{iw}^{+})=W$; t_{fw} is a <u>final</u> or <u>exit</u> time to $I(t)=W$ if $I(t_{fw}^{-})=I(t_{fw})=W$ and $I(t_{fw}^{+})<W$. The definition of t_{i0} and t_{f0} are similar and are obtained by changing W to 0.

2.2 Optimal Policies for Three Possible Cases

There are only three different possible cases for the values of $I(t)$. We shall discuss the optimal policies in cases 1,2,and 3 below.

<u>Case 1</u> $0<I(t)<W$

This implies $\rho_1=\rho_2=0$. We have two different regimes 1A and 1B in this case.

<u>Regime 1A</u> $\lambda(t)<c$. From (6) and (7) we have the following optimal policies;

$$u^*=0, \ p^*=\max\{0;(a/b+\lambda)/2\}, \ I<0 \qquad (12)$$

Regime 1B $\lambda(t)>c$.

$$u^*=\bar{u} \text{ and } p^*=\min\{a/b; (a/b+\lambda)/2\} \tag{13}$$

Case 2 $I(t)=0$ and $\dot{I}(t)=0$.

This implies $\rho_2=0$, $u^*=a-bp^*$, and $(a/b+c)/2\leq p^*<a/b$. We also have regimes 2A and 2B in this case.

Regime 2A $\bar{u}\geq(a-bc)/2$. An application of the maximum principle gives the following optimal policies;

$$\begin{aligned} &u^*=(a-bc)/2,\ p^*=(a/b+c)/2,\ \lambda+p_1=c \text{ and} \\ &\mu_1=\mu_2=\eta_1=\eta_2=0 \end{aligned} \tag{14}$$

Regime 2B $\bar{u}<(a-bc)/2$ and $\dot{a}\leq bh$. The maximum principle give the following optimal policies;

$$\begin{aligned} &u^*=\bar{u},\ p^*=(a-\bar{u})/b,\ \lambda+\rho_1=(a-2\bar{u})/b, \\ &\mu_1=\eta_1=\eta_2=0, \text{ and } \mu_2=(a-2\bar{u}-bc)/b>0 \end{aligned} \tag{15}$$

Note that if both $\bar{u}<(a-bc)/2$ and $\dot{a}>bh$ hold then $\dot{\rho}_1=\dot{a}/b-h>0$, which leads to a contradiction to (11).

Case 3 $I(t)=W$ and $\dot{I}(t)=0$.

These conditions imply $\rho_1=0$, $u^*=a-bp^*$, and $(a/b+c)/2 \leq p^* < a/b$. We have only one regime 3A in this case.

<u>Regime 3A</u> $\bar{u}<(a-bc)/2$ and $\dot{a} \geq bh$. The maximum principle yields;

$$u^*=\bar{u},\ p^*=(a-\bar{u})/b,\ \lambda-\rho_2=(a-2\bar{u})/b,\ \mu_1=\eta_1=\eta_2=0, \tag{15}$$

$$\text{and } \mu_2=(a-2\bar{u}-bc)/b>0$$

Note that if $\bar{u}>(a-bc)/2$ then $u^*=(a-bc)/2$, $p^*=(a/b+c)/2$, and $\lambda-\rho_2=c$. This implies $\dot{\lambda}-\dot{\rho}_2=0$ and contradicts the fact that $\dot{\lambda}-\dot{\rho}_2 \geq h$. Similarly, if $\bar{u}<(a-bc)/2$ and $\dot{a}<bh$ then $\lambda-\rho_2=(a-2\bar{u})/b$, $\dot{\lambda}-\dot{\rho}_2=\dot{a}/b<h$ and this again leads to a contradiction. Therefore Regime 3A is the only case to be considered here.

A simple economic interpretation of Cases 2 and 3 is as follows. It is easy to prove (proof omitted) that $\dot{I}=0$ implies that $(a-bc)/2$ is the best production rate without considering the production capacity. Therefore, we can call $(a-bc)/2$ an <u>ideal perfect production rate</u>. In Regime 2A, $\bar{u} \geq (a-bc)/2$ means that we can set the production rate to be the ideal perfect production rate without violating the production capacity. Thus, we don't need to build up inventory for the future, i.e., Keeping the warehouse empty is our optimal strategy. In Regime 2B, $\bar{u}<(a-bc)/2$ implies that the production rate couldn't reach to the ideal perfect production rate because of a lower production capicity. However, $\dot{a} \leq bh$ hints that the rate of price increase, $\dot{a}/b$, is not greater than the marginal inventory cost, h. Therefore, it does not pay to build up inventory. Similarly, in Regime 3A the production capacity is less than the ideal perfect production rate and the rate of price increase is greater than or equal to the marginal inventory cost. Thus, it pays to build up inventory.

2.3 Theoretical Results

Let us investigate the behavior of I(t). First, we shall establish a sufficiently large initial inventory I_0 such that $I_0 \geq \bar{I}_0$ implies $u^* = 0$ for all t. From Regime 1A, and Equation (9) we know

$$\bar{I}_0 = \int_0^T [a - b\max\{0; [a/b + h(t-T)]/2\}]dt \tag{17}$$

The inventory $\bar{I}_0$ is just sufficient to meet demand when price is determined by either of the first two rules in (7).

Theorem 1 $I_0 \geq \bar{I}_0 \Leftrightarrow I(T) = I_0 - \bar{I}_0 \geq 0$, and $\lambda(T) = 0$

$\Leftrightarrow u^* = 0$ for all t, and $p_*(T) = a/(2b)$.

Proof. We prove the second implication only. Assume $u^* = 0$ and $p^*(T) = a/(2b)$. This can occur only in Regime 1A. Thus, $0 < I(t) < W$, $\lambda(t) = h(t-T)$ for all t, and $I(T) = I_0 - \bar{I}_0 \geq 0$. The verification of the converse is trivial.

Corollary 1 $I_0 < \bar{I}_0 \Leftrightarrow I(T) = 0$ and $\lambda(T) > 0$

<u>Theorem 2</u> We can conclude that $\dot{I}\leq 0$ under each of the following cases:

(i) $I_0 \geq \bar{I}_0$

(ii) $I_0 < \bar{I}_0$ and $\bar{u} \geq (a-bc)/2$ for all t

(iii) $I_0 < \bar{I}_0$; whenever $\bar{u} < (a-bc)/2$ we have $\dot{a} \leq bh$.

<u>Proof</u>. Case (i) follows from Theorem 1. We now prove cases (ii) and (iii). Suppose not, i.e., $\dot{I}(t)>0$ in $(\tau, \tau+\varepsilon)$; see Figure 1. This can happen only in Regime 1B. Thus, for all $t\varepsilon(\tau+\varepsilon/2, \tau+\varepsilon)$ we have $\lambda>c$, $u^{*}=\bar{u}$ and $p^{*}=\min[a/b; (a/b+\lambda)/2]$. Hence

$$\dot{I}=\bar{u}-(a-bp^{*}) = \begin{cases} \bar{u} & \text{if } \lambda \geq a/b \quad (p^{*}=a/b) \\ \bar{u}-(a-b\lambda)/2 & \text{if } \lambda < a/b \quad (p^{*}=(a/b+\lambda)/2) \end{cases}$$

<u>In Case (ii)</u>, the conditions, $\bar{u} \geq (a-bc)/2$ with $\lambda(\tau+\varepsilon)>c$, imply $\bar{u}-(a-b\lambda)/2>\bar{u}-(a-bc)/2\geq 0$ on $(\tau+\varepsilon/2, \tau+\varepsilon)$. Thus $\dot{I}(\tau+\varepsilon)>0$. By repeating this argument for the intervale $(\tau+\varepsilon, \tau+2\varepsilon)$, $(\tau+2\varepsilon, \tau+3\varepsilon)$, ..., we see that $I(T)>0$. This contraditics Corollary 1 which says $I(t)=0$ in Case (ii).

<u>In Case (iii)</u>, $\ddot{I}=0$ or $(bh-\dot{a})/2\geq 0$. Thus, $\dot{I}(\tau+\varepsilon)>0$, which, as above, contradict $I(T)=0$ for Case (iii).

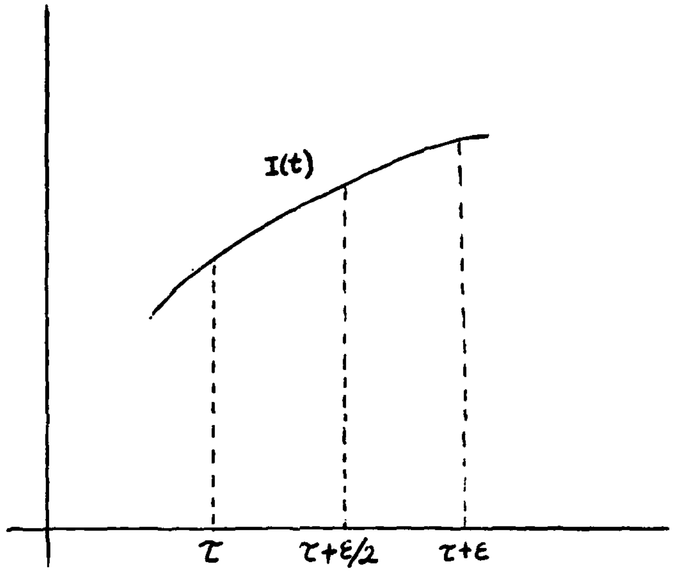

Figure 1

Corollary 2 Under the cases specified in Theorem 2, the optimal path for I(t) is in one of the following three cases:

(a) $I(t)>0$ for all t (This case can happen if and only if $I_0>\bar{I}_0$)

(b) $I(t)=0$ for all t (This case can happen if and only if $I_0=0$)

(c) $I(t)>0$ on $[0,t_1]$ and $I(t)=0$ on $[t_1,T]$ (This case can occur if and only if $0<I_0\leq\bar{I}_0$)

We now discuss how t_1 in Corollary 2(c) is determined. For convenience, let us define the function $\psi(t)=(\lambda+\rho_1-\rho_2)(t)$ which is similar to the ψ function in Pekelman [7]. In Regimes 2A and 2B, we have

$$\psi\equiv\psi_0=\begin{cases} c & \text{if } \bar{u}\geq(a-bc)/2 \\ (a-2\bar{u})/b & \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a}\leq bh \end{cases} \tag{18}$$

or $\psi_0=\max\{c;\ (a-2\bar{u})/b\}$. Intervals such that $\bar{u}\geq(a-bc)/2$ or $[\bar{u}<(a-bc)/2$ and $\dot{a}\leq bh]$ are defined to be <u>empty warehouse transitive intervals</u> on which the boundary condition $I(t)=0$ holds. Similarly, by using the results in (16) for Regime 3A we have

$$\psi\equiv\psi_W=(a-2\bar{u})/b \tag{19}$$

Intervals such that $\bar{u}<(a-bc)/2$ and $\dot{a}\geq bh$ are defined to be <u>full warehouse trasitive intervals</u> on which the boudnary condition $I(t)=W$ holds.

After defining the function ψ_0, we can use a binary search program to find the values of $\lambda(0)$ and t_1 in Corollary 2(c). By Corollary 1 we have $\lambda_0\equiv\lambda(0)>-hT\equiv\underline{\lambda}_0$. If condition (ii) of Theorem 2 occurs then $\psi_0(t_1)=c$ so that $\lambda_0<c$. If condition (iii) of Theorem 2 happens the $\lambda_0<a(0)/b$; otherwise $\lambda(t)>a(t)/b$ (because $\dot{\lambda}=h\geq\dot{a}/b$), $p(t)=a(t)/b$ for all t so that $I(T)>I_0$ and leads to a contradiction. Thus, $\lambda_0<a(0)/b\equiv\bar{\lambda}_0$ holds in any of the three conditions of Theorem 2. To find the value of t_1, we can first

guess $\lambda_0^{(1)}=(\underline{\lambda}_0+\lambda_0)/2$. Check Whether $\lambda_0^{(1)}$ satisfies the following 3 equations or not.

$$\lambda(t_1)=\lambda_0+ht_1=\psi_0(t_1) \tag{20}$$

$$\lambda(t_c)=\lambda_0+ht_c=c \tag{21}$$

$$I_0 = \int_0^{t_c}\{a-b\max[0;(\lambda_0+ht+a/b)/2]\}dt-\int_{t_c}^{t_1}\{\bar{u}-[a-b(\lambda_0+ht)]/2\}dt \tag{22}$$

If yes, then we are done. If the right-hand side of (22) is larger than I_0 then let the new guessing value of λ_0 is $\lambda_0^{(2)}=(\lambda_0^{(1)}+\underline{\lambda}_0)/2$ because $\dot{I}$ is an increasing function with respect to λ_0. Otherwise, let $\lambda_0^{(2)}=(\lambda_0^{(1)}+\bar{\lambda}_0)$. Repeating this binary search procedure we can get the values of t_1 and λ_0. Next, we shall explore the case in which a>bh, i.e., the case in which it pays to store inventory.

<u>Lemma 1</u>. If a is continuous and non-decreasing, then the optimal value of ψ is also continuous and non-decreasing.

<u>Proof</u>. In Regimes 1A and 1B, $\psi=\lambda$ is an increasing continuous function. In Regimes 2A and 2B, $\psi=\psi_0=\max\{c;(a-2\bar{u})/b\}$ is a non-decreasing continuous function. Similarly, in Regime 3A $\psi=\psi_w=(a-2\bar{u})/b$ has the same properties.

<u>Corollary 3</u>. *If a is continuous and non-decreasing then the optimal controls* u^* *and* p^* *are both non-decreasing.*

Proof. It immediately follows from Lemma 1 and (6), and (7).

Corollary 4. If a is continuous and non-decreasing then $u^*(t^0)=\bar{u}$ implies $u^*(t)=\bar{u}$ for all $t\geq t^0$.

Proof. Follows immediately from Corollary 3 and the constraint $u\leq\bar{u}$.

Lemma 2. If $\dot{a}>bh$ then $I(t)=0$ on any interval (t^0,t^1) will imply $u^*=(a-bc)/2<\bar{u}$ on (t^0,t^1).

Proof. $I(t)=0$ on (t^0,t^1) implies this case can happen only in Regime 2A. Thus, $\bar{u}\geq(a-bc)/2$ and $u^*=(a-bc)/2<\bar{u}$ on (t^0,t^1).

From Corollary 4 and Lemma 2, we can easily get the following result.

Corollary 5. If $\dot{a}>bh$ and $u^*(t^0)=\bar{u}$ then $I(t)>0$ almost everywhere on $[t^0,T]$.

Theorem 3. If $\dot{a}>bh$ then there exists an optimal solution such that $I(t)>0$ almost everywhere on $[0,T]$.

Proof. To prove this theorem, we change (6) so that $u^*=\bar{u}$ when $\lambda+\rho_1-\rho_2\geq c$, which comes by putting together the last two parts of (6). If $u^*(t)=0$ for all t then we are in Regime 1A so that $I(t)>0$ almost everywhere on $[0,T]$. Otherwise, there exists $u^*(t^0)=\bar{u}$ for some $t^0\epsilon[0,T]$. Let us define

$$t_0=\inf\{t\mid u^*(t)=\bar{u}\}.$$

If $t_0=0$ then the theorem is true by Corollary 5. If $t_0\neq 0$ and there exists an open interval, say (t_2,t_3) with $t_3\leq t_0$, such that $I(t)=0$ on (t_2,t_3); then $\psi_0(t)\geq c$ and $u^*(t)=\bar{u}$ on (t_2,t_3). This contradicts the fact that $u^*(t)=0$ on (t_2, t_3). Thus, $I(t)>0$ almost everywhere on $[0,t_0]$. By Corollary 5, we also have $I(t)>0$ almost everywhere on $[t_0,T]$. This completes the proof.

Theorem 4. If $I(t)>0$ almost everywhere on $[0,T]$, then the optimal control path for $u^*(t)$ satisfies one of the following three cases:

(I) $u^*=0$ for all t

(II) $u^*=\bar{u}$ for all t

(III) $u^*=0$ for $t\leq t_0$ and $u^*=\bar{u}$ for $t>t_0$

Proof. If $I(t)>0$ almost everywhere on $[0,T]$ then case 2 cannot occur so that there are no jumps in $\lambda(t)$, except in Regime 3A. Regime 3A can occur only following Regime 1B; also 1A cannot follow Regime 1B. Therefore, we have the following three possible optimal policies:

(1) Regime 1A, fc. all t. (This implies (I));

(2) Regime 1B at first, then Regime 3A and 1B maybe alternating several times, at time T the case should be Regime 1B. (This implies (II));

(3) Regime 1A at first, then Regime 1B, after this Regime 3A and 1B maybe alternating several times, at time T the case should be Regime 1B. (This implies (III)).

This completes the proof of Theorem 4.

Here, we shall also discuss the situation in which the exogenous variable $a(t)$ can first go up and then down or first down then up. In fact, using the above results we can easily obtain the following theorems.

<u>Theorem 5</u>. Assume $\bar{t}$ is chosen so that $\dot{a}>bh$ for $t<\bar{t}$ and $\dot{a}\leq bh$ for $t\geq\bar{t}$; i.e., it pays to store production before but not after t. Also assume neither condition (i) nor condition (ii) of Theorem 2 holds. Then there exists an optimal solution such that I(t) satisfies one of the following two cases:

(i) $I(t)>0$ for all $t\varepsilon[0,T)$

(ii) there exists $t_2\geq\bar{t}$ such that $I(t)>0$ on $(0,t_2)$ and $I(t)=0$ on $[t_2,T]$.

<u>Theorem 6</u>. Assume t is chosen so that $\dot{a}\leq bh$ for $t\leq\bar{t}$ and $\dot{a}>bh$ for $t\geq\bar{t}$. Also assume neither condition (i) nor condition (ii) of Theorem 2 holds. Then there exists an optimal solution such that I(t) satisfies one of the following three cases:

(1) $I(t)>0$ for all $t\varepsilon[0,T)$

(2) there exists $t_3\leq\bar{t}$ such taht $I(t)=0$ $[0,t_3]$ and $I(t)>0$ on (t_3,T).

(3) there exists $t_4<t_5\leq\bar{t}$ such that $I(t)>0$ on $[0,t_4)$, $I(t)=0$ on $[t_4,t_5]$, and $I(t)>0$ on (t_5,T).

In general, the interval [0,T] will contain many subintervals which $\bar{u}\geq(a-bc)/2$ or $\dot{a}\leq bh$ or $\dot{a}>bh$. By repeatedly applying the results of Corollary 2 and Theorems 3, 5, and 6 we can construct the solution by piecing together different solutions obtained from application of these theorems.

2.4 Planning Horizon Theorem

The optimal decisions (price and production rate) during the planning period $[0,t^*]$ can be completely independent of the data beyond the forecast horizon $[0,t^{**}]$ with $t^* \leq t^{**}$. That is, no information after t^{**} is required for making optimal decisions on $[0,t^*]$. We therefore call t^* to be a strong planning horizon and t^{**} to be a strong forecast horizon.

In this section, we will explore the strong planning and forecast horizons for the problem.

Lemma 3 Assume we have the optimal trajectory such that t^* and t^{**} are the two consecutive entry times at which inventory hits the boundary contraint. Furthermore, let $I(t^*)=0$ and $I(t^{**})=W$, or $I(t^*)=W$ and $I(t^{**})=0$. Suppose that $t^*<t^{**}$. Then t^* is a strong horizon, and t^{**} is a strong forecast horizon.

Proof. We may assume, without loss of generality, that t^* is the first time to enter the boundary condition $I(t)=0$ or W. Let the optimal value of $\lambda(0)$ be λ_0 to the original problem. Then the values of t^* and λ_0 are decided by the demand in $[0,t^*]$, the production capacity, and the initial inventory as follows.

$$\begin{cases} \lambda_0+ht^*=\lambda(t^{*-})=\psi(t^{*+})=\max\{c;(a-2\bar{u})/b\}, \\ \\ 0(\text{or } W)-I_0=\int_0^{t^*} \dot{I}dt. \end{cases}$$

If we can prove that $\lambda(t)=\lambda_0+ht$ on $(0,t^*)$ is still our optimal solution to any problem having the same information as the original problem on $[0,t^{**}]$ and regardless of its value after t^{**}, then we are done. Here, we will prove the case in which $I(t^*)=W$ and $I(t^{**})=0$, see Figure 2. In fact, using an analogous argument we can prove the other case in which $I(t^*)=0$ and $I(t^{**})=W$.

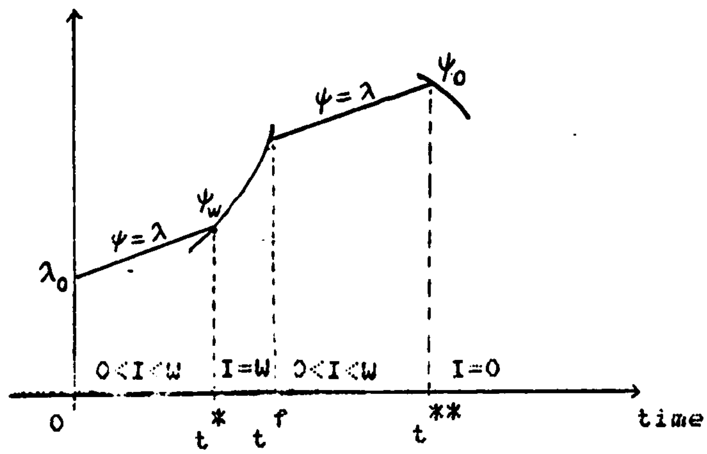

Figure 2

If the new optimal value of $\lambda(0)$ were $\tilde{\lambda}_0$ with $\tilde{\lambda}_0>\lambda_0$, then the new inventory $\hat{I}$ should be greater than the original I on $[0,t^*]$ because the higher value of ψ implies the higher value of I by using (2), (6), and (7). Thus, there exists a $t\epsilon(0,t^*)$ such that $\hat{I}(t)=W$. Let t^0 be the earliest time such that $\hat{I}(t)=W$. Then we have $\hat{I}(t)>0$ for all $t\epsilon(t^0,t^*)$. Otherwise, let t^1 be the first time such that $\hat{I}(t^1)=0$ and $t^1\epsilon(t^0,t^*)$, see Figure 3 for illustration. This implies $0<\hat{I}\leq W$ and $\dot{\hat{\psi}}\geq h=\dot{\psi}$ because Regimes 2A and 2B cannot occur on (t^0,t^1) for $\hat{I}$.

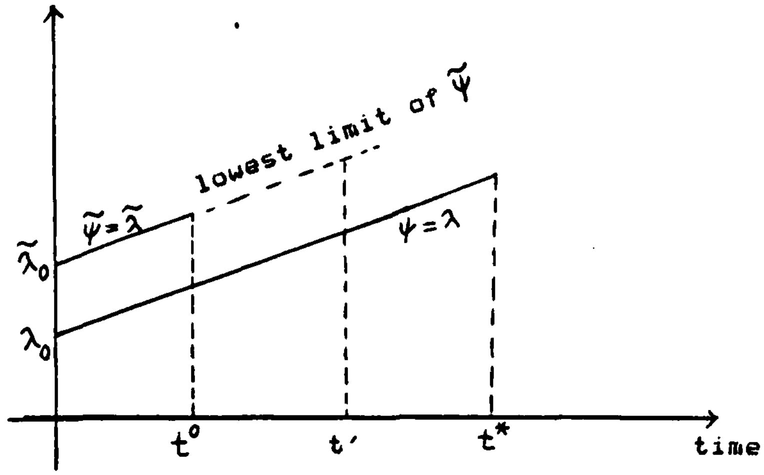

Figure 3

Since $\hat{\psi}(t^0)=\hat{\lambda}_0+ht^0>\lambda_0+ht^0=\psi(t^0)$, $\dot{\hat{\psi}}\geq h=\dot{\psi}$ on $[t^0,t^1]$ we have $\dot{\hat{I}}>\dot{I}$ on $[t^0,t^1]$ and

$$I(t^1)-I(t^0)=\int_{t^0}^{t^1}\dot{I}dt<\int_{t^0}^{t^1}\dot{\tilde{I}}dt=\tilde{I}(t^1)-\tilde{I}(t^0)=-W,$$

or $I(t^1)=I(t^0)-W<0$, which leads to a contradiction. Thus, $\tilde{I}(t)>0$ for all $t\epsilon(t^0.t^*)$. Again, $0<\tilde{I}\leq W$ on (t^0,t^*) implies $\dot{\tilde{\psi}}\geq h=\dot{\psi}$ and $\dot{\tilde{I}}>\dot{I}$ on (t^0,t^*). Then we have

$$\tilde{I}(t^*)-W=\int_{t^0}^{t^*}\dot{\tilde{I}}dt\geq\int_{t^0}^{t^*}\dot{I}dt=W-I(t^0)>0,$$

which again leads to a contradiction to the fact that $\tilde{I}(t)\leq W$ for all t. Therefore, we have shown that $\tilde{\lambda}_0>\lambda_0$ cannot be an optimal solution to the new problem. Next, if the new optimal value of $\lambda(0)$ were $\underset{\sim}{\lambda}_0$ with $\underset{\sim}{\lambda}_0<\lambda_0$, then the new inventory $\underset{\sim}{I}<I$ on $[0,t^*]$ so that $\underset{\sim}{I}(t^*)<W$, see Figure 4. Since $\psi=\psi w$ on (t^*,t^f) where t^f is the final or exit time of $I(t)=W$ we know $\dot{a}>bh$, $\dot{\underset{\sim}{\psi}}\leq h<\dot{\psi}$, $\underset{\sim}{\psi}<\psi$, $\dot{\underset{\sim}{I}}<\dot{I}$, and $\underset{\sim}{I}<I$ on (t^*,t^f). Similarly, we can get $\underset{\sim}{\psi}<\psi$, $\dot{\underset{\sim}{I}}<\dot{I}$, and $\underset{\sim}{I}<W$ on $[t^f,t^{**}]$. This implies

$$\underset{\sim}{I}(t^{**})-\underset{\sim}{I}(t^{f})<I(t^{**})-I(t^{f})=-W,$$

or $\underset{\sim}{I}(t^{**})<0$ and leads to a contradiction again.

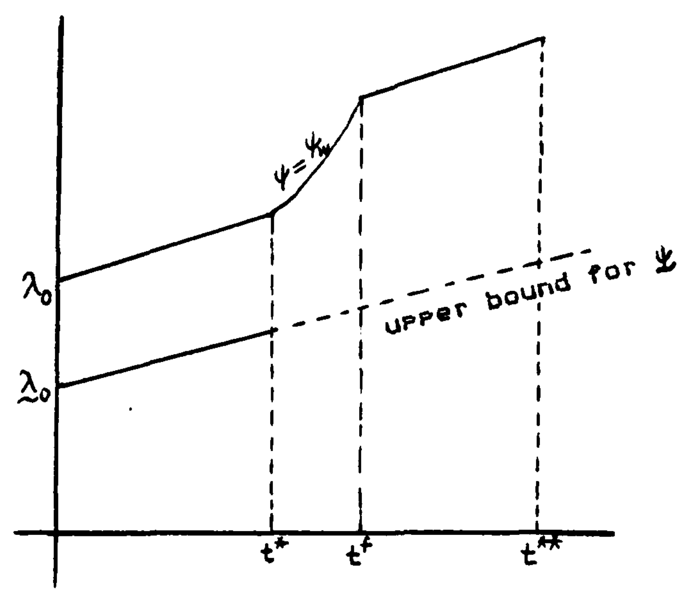

Figure 4

<u>Theorem 7</u>. (Strong Planning Horizon Theorem)

Let t^* and t^f be the entry and exit times to $I(t)=W$ respectively, i.e., $I(t)=W$ for $t\epsilon[t^*,t^f]$, and $I(t^{*-})$ and $I(t^{f+})<W$. Suppose t^{**} is the next entry time to $I(t)=0$. Then all t before t^f are strong planning horizons and all t after t^{**} are strong forecast horizons, i.e. t^f is a maximal strong planning horizon and t^{**} is a mimimal strong forecast horizon. The theorem also holds when 0 and W are interchanged.

<u>Proof</u>. If t^f is not a strong planning horizon then there exists a new optimal solution such that $t^0\epsilon(t^*,t^f)$ is the new exit from $I(t)=W$ boundary. The reasons are as follows. If $t^0<t^*$ then it contradicts to Lemma 3 that t^* is a strong planning horizon. On the other hand, if $t^0>t^f$ then t^f is a strong planning horizon. Using the same arguments as in Lemma 3 we can prove that the new exit time t^0 will lead to a contradiction.

2.5 <u>A Forward Branch and Bound Algorithm</u>

The solution to (1) is obvious if $I_0\geq\bar{I}_0$. Therefore, we may assume, without loss of generality, that $I_0<\bar{I}_0$. In order to find an upper bound to the optimal objective value of (1) we solve the following problem:

$$\max\ \bar{J}_t=\int_t^T\{p(t)[a(t)-bp(t)]-cu(t)\}dt, \tag{23}$$

subject to

$$\dot{I}(t)=u(t)-[a(t)-bp(t)],\ I(t)=I_t \tag{24}$$

It is easy to show that the optimal solutions are given by:

$$p^*(t)=(a+bc)/(2b),\ D^*(t)=(a-bc)/2$$

and

$$\bar{J}_t=\int_t^T(a^2-b^2c^2)/(4b)dt-c[\int_t^T(a-bc)/2dt-I_t] \tag{25}$$

For convenience, we let

$J[\alpha,\beta]=\max\int_\alpha^\beta[p(a-bp)-hI-cu]dt$, subject to constraints (2) and (3).

It is clear that $\bar{J}_t$ in (25) is an upper bound of $J[t,T]$ because we do not have inventory cost and production capacity constraint in problem (23). Next, we shall say that the vertex v_j is <u>fathomed</u> if and only if no further exploration from this vertex can be profitable. Otherwise, we shall say that v_j is <u>unfathomed</u> or <u>alive</u>. We are now in a position to present the algorithm.

<u>Forward Branch and Bound Algorithm</u>

<u>Step 0</u> (Initalization) Begin at the live vertex v_0, where $J=0$. Go to Step 1.

<u>Step 1</u> (Branching) Assume that t_i is the first entry time to the constraint $I(t)=W$ or $I(t)=0$. Solve the following two cases.

<u>Case 1.1</u> Suppose that t_i is the entry time of $I=W$. Then t_i should satisfy the following constraints:

$$\lambda(t_i^-)=[a(t_i))-2\bar{u}]/b \qquad \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a}\geq bh \text{ at } t=t_i \tag{26}$$

$$\lambda(t)=\lambda(t_i^-)+h(t-t_i) \qquad \text{for all } t<t_i \tag{27}$$

$$W-I_0>\int_0^t[u-(a-bp)]dt>-I_0 \quad \text{for all } t<t_i \tag{28}$$

$$W-I_0=\int_0^{t_i}[u-(a-bp)]dt \tag{29}$$

If there exists one or more solutions, then we keep each such solution as successor vertices of v_0 and go to Case 1.2. Otherwise, there are no solutions for this branch, terminate this branch and go to Case 1.2.

Case 1.2 Suppose that t_i is the entry of I=0. The t_i satisifies the same contraints as in Case 1.1, except that (26) and (29) must be replaced by (30) and (31), respectively.

$$\lambda(t_i^-) = \begin{cases} c & \text{if } \bar{u}\geq(a-bc)/2 \text{ at } t=t_i \\ [a(t_i)-2\bar{u}]/b & \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a}\leq bh \text{ at } t=t_i \end{cases} \tag{30}$$

$$-I_0=\int_0^{t_i}[u-(a-bp)]dt \tag{31}$$

Again, we keep all solutions as successor vertices of v_0, if any, and go to Step 2.

Step 2 (Update Bound) Check each new live vertex v_j. If $t^{(j)}=T$ and $J^{(j)}\leq J$, where $t^{(j)}$ is the last time of v_j at which I reaches to W or 0, and $J^{(j)}$ is the objective value of v_j, then the vertex

v_j is fathomed. If $t^{(j)}=T$ and $J^{(j)}>J$, then let $J=J^{(j)}$. Go to Step 3.

<u>Step 3</u> (Fathoming by Bound) Check each new live vertex v_j. If $J[0,t^{(j)}]+\bar{J}_t{}^{(j)}\leq J$, then the vertex v_j is fathomed. Go to Step 4.

<u>Step 4</u> (Branching) If no live vertices exist, go to Step 7; otherwise, select a live vertex v_j. If $I(t^{(j)})=W$, then go to Step 5; otherwise, go to Step 6.

<u>Step 5</u> Assume that t_{fw} is the exit time from $I(t)=W$ and t_i is the next entry time of $I=0$ or $I=W$. As in Step 1, we have two cases.

<u>Case 5.1</u> Suppose t_i is the entry time to $I=0$ and $t_{fw}<t_i$. Solving the following contraints, we can find the values of t_{fw} and t_i.

$$\lambda(t_{fw}+)=[a(t_{fw})-2\bar{u}]/b \qquad \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a}\geq bh \text{ at } t=t_{fw} \tag{32}$$

$$\lambda(t_i^-)=\begin{cases} c & \text{if } \bar{u}\geq(a-bc)/2 \text{ at } t=t_i \\ [a(t_i)-2\bar{u}]/b & \text{if } \bar{u}<(a-bc/2 \text{ and } \dot{a}\leq bh \text{ at } t=t_i \end{cases} \tag{33}$$

$$\lambda(t)=\lambda({}_{fw}{}^{+})+h(t-t_{fw}) \qquad \text{for all } t\epsilon(t_{fw},t_i) \tag{34}$$

$$\bar{u}<(a-bc)/2 \text{ and } a\geq bh \qquad \text{for all } t\epsilon(t^{(j)},t_{fw}) \tag{35}$$

$$-W<\int_{t_{fw}}^{t}[u-(a-bp)]dt<0 \qquad \text{for all } t\epsilon(t_{fw},t_i) \tag{36}$$

$$-W=\int_{t_{fw}}^{t_i}[u-(a-bp)]dt \tag{37}$$

If all above constraints have one or more solutions, then we save them as successor vertices of v_j and go to Case 5.2. Otherwise, terminate this branch and go to Case 5.2.

<u>Case 5.2</u> Let t_i be the entry time to $I(t)=W$. This case is similar to Case 5.1, except that (33) and (37) are replaced by (38) and (39), respectively.

$$\lambda(t_i^-)=[a(t_i)-2\bar{u}/b \qquad \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a}\geq bh \text{ at } t=t_i \tag{38}$$

$$0=\int_{t_{fw}}^{t_i}[u-(a-bp)dt \tag{39}$$

If there exists some solutions to (t_{fw},t_i) in Case 5.1 or Case 5.2, then we keep them as successor vertices of v_j and go to Step 2. Otherwise, v_j is fathomed and to to Step 4.

<u>Step 6</u> Assume that t_{f0} is the exit time from $I(t)=0$, and t_i is the next entry time to $I=0$ or $I=W$, with $t_{f0}<t_i$.

<u>Case 6.1</u> Suppose that t_i is the entry time to $I=0$. Solving the following constraints, we may get the values of t_{f0} and t_i.

$$\lambda(t_{f0}^{+}) = \begin{cases} c & \text{if } \bar{u} \geq (a-bc)/2 \text{ at } t=t_{f0} \\ [a(t_{f0})-2\bar{u}]/b & \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a} \leq bh \text{ at } t=t_{f0} \end{cases} \quad (40)$$

$$\lambda(t_i^{-}) = \begin{cases} c & \text{if } \bar{u} \geq (a-bc)/2 \text{ at } t=t_i \\ [a(t_i)-2\bar{u}]/b & \text{if } \bar{u}(a-bc)/2 \text{ and } \dot{a} \leq bh \text{ at } t=t_i \end{cases} \quad (41)$$

$$\lambda(t)=\lambda(t_{f0}^{+})+h(t-t_{f0}^{+}) \qquad \text{for all } t\epsilon(t_{f0},t_i) \quad (42)$$

$$0<\int_{t_{f0}}^{t_i}[u-(a-bp)]dt<W \qquad \text{for all } t\epsilon(t_{f0},t_i) \quad (43)$$

$$0=\int_{t_{f0}}^{t_i}[u-(a-bp)]dt \quad (44)$$

$$\bar{u} \geq (a-bc)/2 \text{ or } [\bar{u}<(a-bc)/2 \text{ and } \dot{a} \leq bh] \text{ for all } t\epsilon(t^{(j)},t_{f0}) \quad (45)$$

We keep all solutions as successors of v_j, if any, and go to Case 6.2.

<u>Case 6.2</u> Suppose that t_i is the entry time to I=W. Again, this case is similar to Case 6.1, except that we replace (41) and (44) by (46) and (47), respectively.

$$\lambda(t_i^{-})=[a(t_i)-2\bar{u}]/b \qquad \text{if } \bar{u}<(a-bc)/2 \text{ and } \dot{a} \geq bh \text{ at } t=t_i \quad (46)$$

$$W=\int_{f0}^{t_i}[u-(a-bp)]dt \quad (47)$$

If there exist any solutions to (t_{f0},t_i) in Case 6.1 or Case 6.2 then save them and go to Step 3. Otherwise, check whether (45) is satisfied by setting $t_{f0}=T$. If yes, let $I(t)=0$ on $[t^{(j)},T]$ be a feasible solution and go to Step 2. If not, v_j is fathomed and go to Step 4.

Step 7 (Termination) J is optimal.

To illustrate this algorithm, we will solve a simple numerical example.

An Example

Suppose that $T=10$, $a(t)=30+10t-t^2$, $b=1$, $h=1$, $c=10$, $\bar{u}=10$, $W=125/12$ and $I_0=102/12$. We then have that $\bar{u}<(a-bc)/2$ for $t\epsilon(0,10)$, $\dot{a}\geq bh$ for $t\leq 4.5$ and $\dot{a}<bh$ for $t>4.5$.

Step 0 J=0

Step 1 Solving Case 1.1, we have

$$\lambda(t_i^-)=10+10t_i-t_i^2,\ t_i\leq 4$$

$$\lambda(t)=10+9t_i+t-t_i^2,\ \text{for all } t<t_i$$

$$W-I_0=4=\int_0^{t_i}[u-(a-bp)]dt$$

which gives a solution $t_i=1$, $\lambda(t)=18+t$ for $t\leq 1$. Let the vertex corresponding this case is v_1. Next, we find that there are no feasible solutions to Case 1.2, and go to Step 2.

<u>Step 2</u> $t_i^{(1)}=1\neq 10$ and go to Step 3.

<u>Step 3</u> $J[0,1]+\bar{J}_1>0$ so that v_1 is alive. Go to Step 4.

<u>Step 4</u> There is a unique live vertex v_1 with $I(1)=W$. Go to Step 5.

<u>Step 5</u> Solving (32)-(37) simultaneously, we obtain $t_{fw}=2$ and $t_i=7$. Let v_2 be the vertex corresponding to this case. Again, we find that there are no feasible solutions to Case 5.2. Go to Step 2.

<u>Step 2</u> $t_i^{(2)}=7\neq 10$ and go to Step 3.

<u>Step 3</u> $J[0,7]+\bar{J}_7>0$ so that v_2 is still alive. Go to Step 4.

<u>Step 4</u> There exists only one live vertex v_2, and $I(t_i^2)=0$. Go to Step 6.

<u>Step 6</u> There are no feasible solutions to both Case 6.1 and 6.2 since $\bar{u}<(a-bc)/2$ and $\dot{a}\leq bh$ for all $t\geq 7$. We know that $I(t)=0$ on [7,10] is a feasible solution, let v_3 be the corresponding vertex to this, and go to Step 2.

<u>Step 2</u> $t_i^{(3)}=10.$

$$J[0,10]=\int_0^{10}[p(a-bp)-hI-cu]dt,$$

where

$$\psi(t)=\begin{cases}18+t & \text{if } 0\le t\le 1\\ 10+10t-t^2 & \text{if } 1\le t\le 2\\ 24+t & \text{if } 2\le t\le 7\\ 10+10t-t^2 & \text{if } 7\le t\le 10\end{cases}$$

$$u(t)=10 \text{ for all } 0\le t\le 10$$

$$p(t)=\begin{cases}24+5.5t-0.5t^2 & \text{if } 0\le t\le 1\\ 20+10t-t^2 & \text{if } 1\le t\le 2\\ 27+5.5t-0.5t^2 & \text{if } 2\le t\le 7\\ 20+10t-t^2 & \text{if } 7\le t\le 10\end{cases}$$

and

$$I(t)=\begin{cases}102/12+4t-9t^2/4+t^3/6 & \text{if } 0\le t\le 1\\ 125/12 & \text{if } 1\le t\le 2\\ 49/12+7t-9t^2/4+t^3/6 & \text{if } 2\le t\le 7\\ 0 & \text{if } 7\le t\le 10\end{cases}$$

Go to Step 3.

<u>Step 3</u> $J=J[0,10]$ so that v_3 is fathomed. Go to Step 4.

<u>Step 4</u> There are no live vertices. Go to Step 7.

Step 7 $J=J[0,10]$ is the optimal solution. This shows that $t=2$ is a maximal strong planning horizon and $t=7$ is a minimal strong forecast horizon, see Figure 5.

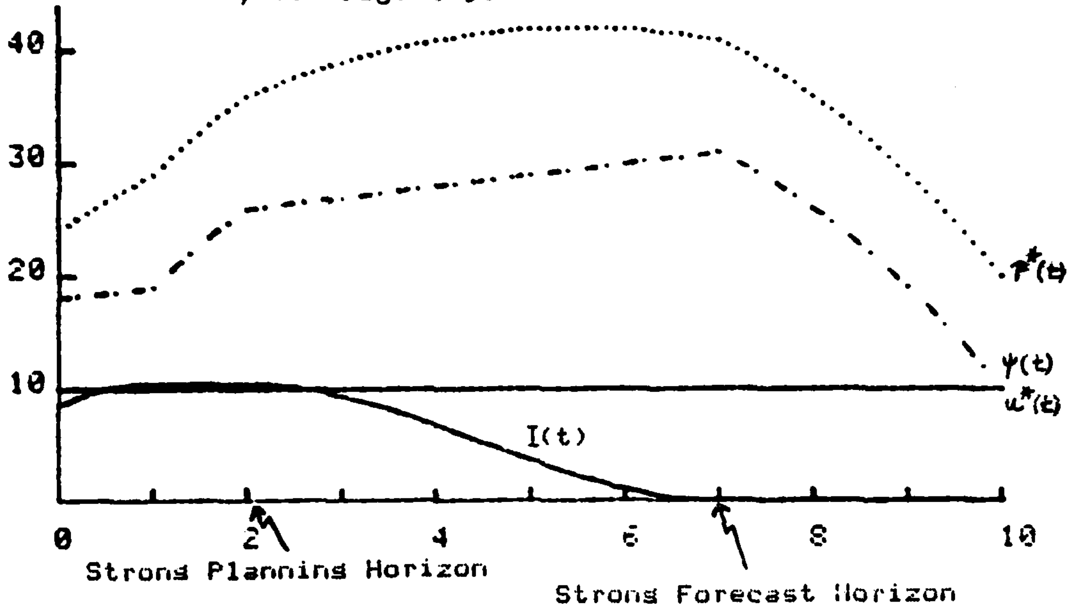

Figure 5. Trajectory for Example.

3. Strictly Convex Non-negative Production Cost Case

In this section, we extend Pekelman's model [7] by considering the production and warehouse capacities. We will see that the optimal inventory trajectories and the strong planning and strong forecast horizons for the linear production cost and strictly convex production cost cases are quite similar.

Suppose that monopolist faces a strictly convex non-negative production cost $f(u)$ instead of a linear production cost cu. Then the monopolist will attempt to maximize the mathematical expression

$$J_2=\int_0^T\{p(a-bp)-hI-f(u)\}dt \qquad (48)$$

subject to (2) and (3). Here, we assume that $f(u)$ is twice differentiable, and $f'(0) \leq a/b$ for all t.

The necessary conditions for an optimal solution to (48) are similar to those of (1) except that Equation (6) is replaced by the following equation

$$u^* = \begin{cases} 0 & \text{if } \lambda+\rho_1-\rho_2 \leq f'(0) \\ g(\lambda+\rho_1-\rho_2) & \text{if } f'(0)<\lambda+\rho_1-\rho_2<f'(\bar{u}) \\ \bar{u} & \text{if } f'(\bar{u}) \leq \lambda+\rho_1-\rho_2 \end{cases} \tag{49}$$

where

$$g=(f')^{-1}. \tag{50}$$

3.1 Optimal Policies

Again, there are only three different possible cases for the values of $I(t)$ in this problem but the situation of optimal policies is slightly different to the linear production case.

Case 1 $0<I(t)<W$. This implies $\rho_1=\rho_2=0$. We have three regimes 1a, 1b, and 1c in this case.

Regime 1a. $\lambda(t)<f'(0)$. We yield

$$u^*=0, \; p^*=\max\{0;(a/b+\lambda)/2\}, \; \dot{I}<0 \tag{51}$$

Regime 1b. $f'(0)<\lambda(t)<g'(\bar{u})$. We obtain

$$u^*=g(\lambda)<0,\ p^*=\min\{a/b;\ (a/b+\lambda)/2\} \tag{52}$$

Regime 1c. $f'(\bar{u})<\lambda(t)$. We get

$$u^*=\bar{u},\ p^*=\min\{a/b;\ (a/b+\lambda)/2\} \tag{53}$$

Case 2 $I(t)=0$ and $\dot{I}(t)=0$. This implies $\rho_2=0$, $u^*=a-bp^*$, and

$$\dot{I} = \begin{cases} g(\lambda+\rho_1)-[a-b(\lambda+\rho_1)]/2 & \text{if } \lambda+\rho_1\leq\min\{f'(\bar{u});\ a/b\} \\ \bar{u}-[a-b(\lambda+\rho_1)]/2 & \text{if } f'(\bar{u})<\lambda+\rho_1<a/b \\ g(\lambda+\rho_1)>0 & \text{if } a/b<\lambda+\rho_1<f'(\bar{u}). \end{cases} \tag{54}$$

Thus, we have two regimes 2a and 2b in this case.

Regime 2a. $\psi_0\equiv\lambda+\rho_1-\rho_2\leq\min\{f'(\bar{u});\ a/b\}$. We have

$$u^*=g(\psi_0) \text{ and } p^*=(a/b+\psi_0)/2 \tag{55}$$

and ψ_0 satisfies the following equation (56) and constraint (57):

$$g(\psi_0)-a/2+b\psi_0/2=0 \tag{56}$$

and $\psi_0=\lambda+\rho_1\leq h$, or

$$\psi_0=a/[b+2g'(\psi_0)]\leq h \tag{57}$$

Regime 2b. $\psi_0\equiv+\rho_1-p_2=(a-2\bar{u})/b>f'(\bar{u})$ and $\dot{a}\leq bh$.

We get $u^*=\bar{u}$, $p^* = (a-\bar{u})/b$, $\mu_1=n_1=n_2=0$, and

$$\mu_2=(a-2\bar{u}-bc)/b>0 \tag{58}$$

Case 3 $I(t)=W$ and $\dot{I}(t)=0$. This implies $\rho_1=0$, $u^*=a-bp^*$. We also have two regimes 3a and 3b in tihs case.

Regime 3a. $\psi_W=\lambda+\rho_1-\rho_2\leq\min\{f'(\bar{u});\ a/b\}$ and $\dot{a}\geq bh$.

$$\text{We obtain } u^*=g(\psi_W),\ p^* = (a/b+\psi_W)/2, \tag{59}$$

and ψ_W satisfies the following constraints:

$$g(\psi_W)-a/2+b\psi_W/2=0 \tag{60}$$

and

$$\psi_W=a/[b+2g'(\psi_W)]\geq h \tag{61}$$

Regime 3b. $\psi_W=\lambda+\rho_1-\rho_2=(a-2\bar{u})/b>f'(\bar{u})$ and $\dot{a}\geq bh$.

$$\begin{aligned}&\text{We yield } u^*=\bar{u},\ p^*=(a-\bar{u})/b,\ \mu_1=n_1=n_2=0, \text{ and}\\ &\mu_2=(a-2\bar{u}-bc)/b>0\end{aligned} \tag{62}$$

Note that Case 3 can occur only if $\dot{a}\geq bh$. That is, full warehouse can happen only if the rate of price increase is not less than the inventory cost.

3.2 Theoretical Results and Planning Horizon Theorem

Using argument similar to that in Section 2 it is clear that Theorem 1 is still the in the strictly convex production cost case but Theorem 4 is not correct in that case. We also need a slight modification of the arguments to get results similar to those in Theorem2 2 and 3.

Theorem 8. If $I_0 \geq \bar{I}_0$ or $\dot{a}/b \leq h$, then $\dot{I} \leq 0$ for all t.

Proof. If $I_0 \geq \bar{I}_0$ then the proof is trivial. Suppose that $I_0 < \bar{I}_0$ and there exists an open interval $(\tau, \tau+\varepsilon)$ such that $\dot{I}(t) > 0$ on $(\tau, \tau+\varepsilon)$. Then this can occur only in Regimes 1b and 1c. Therefore, we have

$$\dot{I} = \begin{cases} g(\lambda)-(a-b\lambda)/2 & \text{if } f'(0)<\lambda<\min\{a/b;\ f'(\bar{u})\} \\ g(\lambda)>0 & \text{if } a/b<\lambda<f'(\bar{u}\_ \\ \bar{u}-(a-b\lambda)/2 & \text{if } f'(\bar{u})<\lambda<a/b \\ \bar{u}>0 & \text{if } \lambda>\max\{a/b;\ f'(\bar{u})\} \end{cases}$$

and

$$\ddot{I} = \begin{cases} (bh-a)/2+g'(\lambda)h & \text{if } f'(0)<\lambda<\min\{a/b;\ f'(\bar{u})\} \\ & \\ (bh-a)/2 & \text{if } f'(\bar{u})<\lambda<a/b \end{cases}$$

Since f(u) is strictly convex and non-negative, we know f' and $g-(f')^{-1}$ are increasing. Hence, g' must be positive. So, $I(\tau+\varepsilon)>0$ which also implies $I(T)>0$. This contradicts that fact tht $I_0<I_0$.

<u>Theorem 9</u>. If $\dot{a}/[b+2g']>h$ for all t, then $I(t)>0$ almost everywhere on $[0,T]$.

<u>Proof</u>. If not, $I(t)=0$ on some open interval $(\tau, \tau+\varepsilon)$. Since g' is positive, we have $\dot{a}/b>\dot{a}/[b+2g']>h$ and Regimes 2a and 2b therefore cannot happen. This leads to a contradiction.

By using a proof essentially identical to that of Theorem 7, we can show that the same Planning Horizon Theorem which holds for the linear cost case (Theorem 7) also holds for the convex cost case. We restate the theorem for completeness.

<u>Theorem 10</u>. (Strong Planning Horizon Theorem)

Let t^f be the exit time from $I(t)=W$ and t^{**} be the next entry time to $I(t)=0$. Then t^f is a maximal strong planning horizon and t^{**} is a minimal strong forecast horizon. The theorem also holds when 0 and W are interchanged.

4. Conclusion

In this paper we have studied a general price-production-inventory model with linear and nonlinear production costs. We characterized the optimal trajectories and showed that there could be strong planning and forecast horizons. We presented a forward branch and bound algorithm which identifies strong planning and forecast horizons, and uses them to decompose the problem into a set of smaller problems. The algorithm is illustrated by means of a simple example.

It would be possible to extend the results of this paper in several different ways. For instance, more general demand functions could be considered. In a subsequent paper we intend to study the model when backlogging is permitted. Also, if we change the linear inventory cost into a strictly increasing non-negative inventory holding cost, then our strong planning horizon theorem is still true. The other theorems in this paper, with suitable modifications, would also be true.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER MSRR No. 476	2. GOVT ACCESSION NO. AD-A104 083	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Strong Planning and Forecast Horizons for a Model with Simultaneous Price and Production Decisions		5. TYPE OF REPORT & PERIOD COVERED Technical Report June 1981
		6. PERFORMING ORG. REPORT NUMBER MSRR No. 476
7. AUTHOR(s) Gerald L. Thompson Suresh P. Sethi Jinn-Tsair Teng		8. CONTRACT OR GRANT NUMBER(s) N00014-78-C-0340
9. PERFORMING ORGANIZATION NAME AND ADDRESS Graduate School of Industrial Administration Carnegie-Mellon University Pittsburgh, Pennsylvania 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047-048
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 434) Arlington, Virginia 22217		12. REPORT DATE June 1981
		13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Strong planning horizon, Optimal control theory, Price-production-inventory model, Forward branch and bound algorithm.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we have solved a general inventory model with simultaneous price and production decisions. Both linear and non-linear (strictly convex) production cost cases are treated. Upper and lower bounds are imposed on state as well as control variables. The problem is solved by using the Lagrangian form of the maximum principle. Strong planning and strong forecase horizons are obtained. These arise when the state variable reaches its upper or lower bound. The existence of these horizons permits the decomposition of the whole problem into a set of smaller problems, which can be

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) separately, and their solutions put together to form a complete solution of the problem. Finally, we derive a forward branch and bound algorithm to solve the problem. The algorithm is illustrated with a simple example.